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LETTER TO THE EDITOR

Exact solution for the quantum Davey–Stewartson I system with time-dependent applied forces

G D Pang†‖, F C Pu‡ and B H Zhao§

† Department of Mathematics, UMIST, Manchester M60 1QD, UK
‡ Institute of Physics, Chinese Academy of Sciences, Beijing, People's Republic of China
§ Graduate School, Chinese Academy of Sciences, Beijing, People's Republic of China

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Abstract. The quantized (2+1)-dimensional Davey–Stewartson I system with time-independent applied forces has been solved in a recent letter. In this letter, we give the solution for the case when the applied forces are time-dependent.

The Hamiltonian of the system considered here is

$$H=\int d\xi\, d\eta \frac{1}{2}\left\{-q^*(\partial_\xi^2+\partial_\eta^2)q+\frac{c}{2}q^*[(\partial_\xi\partial_\eta^{-1}+\partial_\eta\partial_\xi^{-1})(q^*q)]q+(u_1+u_2)q^*q\right\} \tag{1}$$

where q and q^* are field operators which satisfy the following equal-time commutation relations

$$\begin{aligned}&[q(\xi,\eta,t),q^*(\xi',\eta',t)]=2\delta(\xi-\xi')\delta(\eta-\eta')\\ &[q(\xi,\eta,t),q(\xi',\eta',t)]=0\end{aligned} \tag{2}$$

and ∂_η^{-1} is defined by

$$\partial_\eta^{-1}(q^*q)=\frac{1}{2}\left(\int_{-\infty}^{\eta}d\eta'-\int_{\eta}^{\infty}d\eta'\right)(q^*(\xi,\eta',t)q(\xi,\eta',t)). \tag{3}$$

In (1) c is the coupling constant, and $u_1\equiv u_1(\xi,t)$ and $u_2\equiv u_2(\eta,t)$ are time-dependent applied potentials (here we assume they are all real).

We note that the Heisenberg equation $i\partial_t q=[q,H]$ for the field operator q is just the Davey–Stewartson I (DSI) equation [2]

$$i\partial_t q=-\tfrac{1}{2}[\partial_x^2+\partial_y^2]q+iA_1q-iA_2q \tag{4}$$

where A_1 and A_2 now are chosen as

$$\begin{aligned}A_1&=-ic\partial_\xi\partial_\eta^{-1}(q^*q)-iu_1(\xi,t)\\ A_2&=ic\partial_\eta\partial_\xi^{-1}(q^*q)+iu_2(\eta,t)\end{aligned} \tag{5}$$

and $x=(1/2)(\xi+\eta)$, $y=(1/2)(\xi-\eta)$. Solutions of the initial-boundary-value problem for the related classical DSI equation can be found in [3].

‖ On leave of absence from Institute of Physics, Chinese Academy of Sciences, Beijing, People's Republic of China.

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In the case when u_1 and u_2 are time-independent, the exact solution for the Hamiltonian (1) has been given in [1]. Here we consider the case when u_1 and u_2 are time-dependent, and we are to find state $|\psi\rangle$ which satisfies

$$H|\psi\rangle = \mathrm{i}\partial_t|\psi\rangle \tag{6}$$

with the initial condition $|\psi(t=0)\rangle = |\psi_0\rangle$.

Now consider an N-particle state

$$|\psi\rangle = \int \mathrm{d}\xi_1\,\mathrm{d}\eta_1 \ldots \mathrm{d}\xi_N\,\mathrm{d}\eta_N \psi(\xi_1, \eta_1; \ldots; \xi_N, \eta_N; t) q^*(\xi_1, \eta_1) \ldots q^*(\xi_N, \eta_N)|0\rangle \tag{7}$$

where $|0\rangle$ is the vacuum state with the property $q|0\rangle = 0$, and $q^*(\xi, \eta) = \mathrm{e}^{-\mathrm{i}Ht} q^*(\xi, \eta, t)\,\mathrm{e}^{\mathrm{i}Ht}$ is the creation operator in the Schrödinger picture. Substituting (7) into (6), we obtain the following time-dependent Schrödinger equation for the wavefunction ψ

$$-\sum_{i=1}^{N} (\partial^2_{\xi_i} + \partial^2_{\eta_i})\psi + c\sum_{i<j} [\delta'(\xi_{ij})\varepsilon(\eta_{ij}) + \delta'(\eta_{ij})\varepsilon(\xi_{ij})]\psi + \sum_{i=1}^{N} [u_1(\xi_i, t) + u_2(\eta_i, t)]\psi$$
$$= \mathrm{i}\partial_t\psi \tag{8}$$

where $\eta_{ij} = \eta_i - \eta_j$, $\delta'(\eta_{ij}) = \partial_{\eta_i}\delta(\eta_{ij})$ and $\varepsilon(\eta_{ij}) = 1$ for $\eta_{ij} > 0$, 0 for $\eta_{ij} = 0$, and -1 for $\eta_{ij} < 0$. Because of the commutation relations (2), the N-particle wavefunction ψ should be symmetric with respect to the permutation of coordinates (ξ_i, η_i).

In order to solve (8) with the initial condition $\psi|_{t=0} = \psi_0$, we suppose that $\{X_l(\xi, t)\}$ and $\{Y_k(\eta, t)\}$ are two sets of orthonormal wavefunctions for the following 1D time-dependent Schrödinger equations

$$\begin{aligned} [-\partial^2_\xi + u_1(\xi, t)]X_l(\xi, t) &= \mathrm{i}\partial_t X_l(\xi, t) \\ [-\partial^2_\eta + u_2(\eta, t)]Y_k(\eta, t) &= \mathrm{i}\partial_t Y_k(\eta, t) \end{aligned} \tag{9}$$

where $l(k)$ is the index used to distinguish different wavefunctions. It can be easily proved by directly checking that

$$\psi = \sum_{\{l_i, k_i\}} A(\{l_i\}, \{k_i\})\varphi_{\{l_i\}\{k_i\}} \tag{10}$$

is the solution of the time-dependent Schrödinger equation (8), where A is a time-independent coefficient determined by the initial value of ψ and $\varphi_{\{l_i\}\{k_i\}}$ is defined by

$$\varphi_{\{l_i\}\{k_i\}} = \left\{\prod_{i<j}\left[1 - \frac{c}{4}\varepsilon(\xi_{ij})\varepsilon(\eta_{ij})\right]\right\}\left[\sum_P (-1)^P \prod_{i=1}^{N} X_{l_i}(\xi_{Pi}, t)\right]$$
$$\times\left[\sum_P (-1)^P \prod_{i=1}^{N} Y_{k_i}(\eta_{Pi}, t)\right]. \tag{11}$$

Here P are $N!$ permutations, and $\{l_i\}$ and $\{k_i\}$ are two ordered sets with $l_1 < l_N$ and $k_1 < k_N$.

In order to determine A, we need the following expression for the scalar product between φ^* and φ

$$(\varphi^*_{\{l_i\}\{k_i\}}, \varphi_{\{l'_i\}\{k'_i\}}) \equiv \int \mathrm{d}\xi_1\,\mathrm{d}\eta_1 \ldots \mathrm{d}\xi_N \mathrm{d}\eta_N \varphi^*_{\{l_i\}\{k_i\}}\varphi_{\{l'_i\}\{k'_i\}}$$
$$= (N!)^2 \frac{I(c)}{I(0)} \delta_{\{l'_i\}\{l_i\}}\delta_{\{k'_i\}\{k_i\}} \tag{12}$$

where

$$I(c)=\int \mathrm{d}\xi_1\,\mathrm{d}\eta_1\ldots\mathrm{d}\xi_N\,\mathrm{d}\eta_N \prod_{i<j}\left[1-\frac{c}{4}\varepsilon(\xi_{ij})\varepsilon(\eta_{ij})\right]^2 \tag{13}$$

and $\delta_{\{l_i\}\{l'_i\}}=1$ for the case when the two set $\{l_i\}$ and $\{l'_i\}$ are the same and 0 for the other cases. (12) is obtained by using the symmetric property of φ and the orthonormal properties of $\{X_l\}$ and $\{Y_k\}$.

With (12) we can derive an expression for A. Substituting it into (10) we finally obtain that

$$\psi=\sum_{\{l_i\}\{k_i\}}(N!)^{-2}\frac{I(0)}{I(c)}(\varphi^*_{\{l_i\}\{k_i\}}|_{t=0},\psi_0)\varphi_{\{l_i\}\{k_i\}} \tag{14}$$

is the solution of the time-dependent Schrödinger equation (8) with the initial condition $\psi|_{t=0}=\psi_0$.

Physical quantities such as the transition probability of the system can be readily calculated by using (14), provided the time-dependent 1D Schrödinger equations (9) have been solved. Although the spectral theory of (9) for general potentials u_i still remains open, there do exist some classes of potentials u_i for which (9) has been solved. For example, when

$$u_1(\xi,t)=\omega_1(\xi-v_1t-\xi_0)^2/2 \qquad u_2(\eta,t)=\omega_2(\eta-v_2t-\eta_0)^2/2 \tag{15}$$

with $\omega_1(\omega_2)>0$ (here ω_i and v_i are all constants), the solutions of (9) are

$$\begin{aligned} X_l(\xi,t)&=H_l(\lambda_1\hat{\xi})\exp\{\mathrm{i}\tfrac{1}{2}v_1(\xi-\tfrac{1}{2}v_1t-\xi_0)-\mathrm{i}\sqrt{2\omega_1}\,(l+\tfrac{1}{2})t-\tfrac{1}{2}\lambda_1^2\hat{\xi}^2\}\\ Y_k(\eta,t)&=H_k(\lambda_2\hat{\eta})\exp\{\mathrm{i}\tfrac{1}{2}v_2(\eta-\tfrac{1}{2}v_2t-\eta_0)-\mathrm{i}\sqrt{2\omega_2}\,(k+\tfrac{1}{2})t-\tfrac{1}{2}\lambda_2^2\hat{\eta}^2\}\end{aligned} \tag{16}$$

where

$$\lambda_i\equiv(\omega_i/2)^{1/4} \qquad \hat{\xi}\equiv\xi-v_1t-\xi_0 \qquad \hat{\eta}\equiv\eta-v_2t-\eta_0$$

and $H_l(\xi)$'s are the Hermite polynomials. This is the simplest example. Some complicated examples can be found in [4] and [5]. The problem of finding new classes of potentials u_i for which (9) can be solved will be further studied.

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